Math 756 • Midterm Exam • Victor Matveev
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1) (20pts) Show that the image of the unit disk under the transformation $z = \dfrac{2\zeta}{\zeta - 1}$ is a half-plane. Next, make a rough guess about the image of a square inscribed into the unit circle, with vertices at ±1 and ±*i*, using only the information about the mapping of the vertices of this square. Sketch your guess. Keep in mind that this map is conformal.

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2) (20pts) We learned that $\sum_{n=1}^{\infty} \dfrac{1}{n^k}$ can be calculated by contour integration of an analytic function $f(z) = \pi \cot \pi z / z^k$ over a rectangular contour with vertices at ±(*N*+½) ± *i N*, in the limit $N \to \infty$. Use a similar method to calculate $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n^2}$ (hint: change the integrand slightly). Make sure to comment on the boundedness of each piece of the resulting contour integral.

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3) (30pts) Solve the following Volterra integral equation. You may use without derivation the Laplace transform result $L[\sin \omega t] = \dfrac{\omega}{s^2 + \omega^2}$; denote $\gamma = \omega(\omega - 1)$ to simplify algebra. Does solution exist when $\omega = +1$?

$$y(t) = t + \int_0^t y(t - \tau) \sin \omega\tau \, d\tau$$

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4) (30pts) Use the Fourier transform to write down the solution for the following modification of the heat (or diffusion) equation, as a convolution of the boundary condition with the Green's function of this problem. Make a rough sketch of the Green's function *g*(*x*, *t*) for any two different values of time, t_1 and $t_2 > t_1$. Comment on the physical meaning of parameter ω.

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} - \omega \dfrac{\partial u}{\partial x}; \quad -\infty < x < +\infty, \quad t > 0, \ \omega > 0 \\ u(x, t = 0) = f(x) \end{cases}$$